

Code-Based Cryptography

Error-Correcting Codes and Cryptography

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1. Introduction I - Cryptography
2. Introduction II - Coding Theory
3. Encoding (Linear Transformation)
4. Parity Checking
5. Error Correcting Capacity
6. Decoding (A Difficult Problem)
7. **Reed-Solomon Codes**
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Generalized Reed-Solomon codes

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$$\begin{array}{l} \text{ev}_{\mathbf{a}, \mathbf{b}} \quad L_k \longrightarrow \mathbb{F}_q^n \\ f(X) \longmapsto \mathbf{b} * f(\mathbf{a}) \\ \qquad \qquad \qquad = (b_1 f(a_1), \dots, b_n f(a_n)) \end{array}$$

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Parameters of GRS codes

Proposition

The $\text{GRS}_k(\mathbf{a}, \mathbf{b})$ is an $[n, k]_q$ code with minimum distance $d = n - k + 1$

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$$G = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{d-2} & a_2^{d-2} & \dots & a_n^{d-2} \end{pmatrix} \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_n \end{pmatrix}$$

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$$= \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ b_1 a_1 & b_2 a_2 & \dots & a_n \\ b_1 a_1^2 & b_2 a_2^2 & \dots & b_n a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1 a_1^{d-2} & b_2 a_2^{d-2} & \dots & b_n a_n^{d-2} \end{pmatrix} \in \mathbb{F}_q^{k \times n}$$

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Proposition

$$\text{GRS}_k(\mathbf{a}, \mathbf{b})^\perp = \text{GRS}_{n-k}(\mathbf{a}, \mathbf{b}')$$

Decoding GRS codes

Let $\mathcal{C} = \text{GRS}_k(\mathbf{a}, \mathbf{b})$ be an $[n, k, d]_q$ code with parity check matrix:

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{d-2} & a_2^{d-2} & \cdots & a_n^{d-2} \end{pmatrix} \cdot \begin{pmatrix} b_1 & b_2 & \cdots & 0 \\ 0 & & & b_n \end{pmatrix} \in \mathbb{F}_q^{(n-k) \times n}$$

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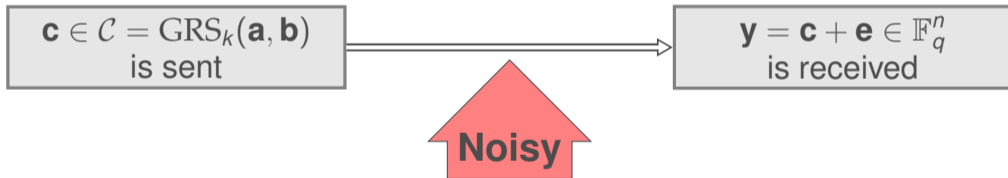
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$\mathbf{y} = \mathbf{c} + \mathbf{e} \in \mathbb{F}_q^n$
is received



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Error positions:

$$I = \{i \in \{1, \dots, n\} \mid b_i f(a_i) \neq y_i\} = \{i_1, \dots, i_t\}$$

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→ If $i \in I$, $E(a_i) = 0$

→ Otherwise, $b_i f(a_i) = y_i$

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This system has solution if $2t + k < n$

Thus, we can correct up to $t < \frac{n-k}{2} = \frac{d-1}{2}$

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