Code-Based Cryptography

Message Attacks (ISD)

Nicolas Sendrier

Code-Based Cryptography

1. Error-Correcting Codes and Cryptography
2. McEliece Cryptosystem
3. **Message Attacks (ISD)**
4. Key Attacks
5. Other Cryptographic Constructions Relying on Coding Theory

3. Message Attack (ISD)

1. **From Generic Decoding to Syndrome Decoding**
2. Combinatorial Solutions: Exhaustive Search and Birthday Decoding
3. Information Set Decoding: the Power of Linear Algebra
4. Complexity Analysis
5. Lee and Brickell Algorithm
6. Stern/Dumer Algorithm
7. May, Meurer, and Thomae Algorithm
8. Becker, Joux, May, and Meurer Algorithm
9. Generalized Birthday Algorithm for Decoding
10. Decoding One Out of Many

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cryptogram (or ciphertext) → y

message (or cleartext) → x

public key → G

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cryptogram (or ciphertext) — message (or cleartext) — public key — random error

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The adversary knows the cryptogram and the public key
and wishes to recover the message (or equivalently the error)

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Only an arbitrary generator matrix is known

$\rightarrow$ **generic decoding problem**

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In contrast with the usual situation where the code is known in advance, a generic decoder takes a q-ary linear $[n, k]$ code as argument

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“small” = of small Hamming weight

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Generic Syndrome Decoder:

$$\begin{array}{rccc} \Psi : & \mathbf{F}_q^{n-k} \times \mathbf{F}_q^{(n-k) \times n} & \to & \mathbf{F}_q^n \\ & (s, H) & \mapsto & e \end{array}$$

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Those two kinds of decoders are equivalent

$\rightarrow$ **we will consider only syndrome decoding**

The Syndrome Decoding Problem

Syndrome Decoding Problem

Instance: $H \in \{0,1\}^{(n-k)\times n}$, $s \in \{0,1\}^{n-k}$, an integer $w > 0$

Answer: $e \in \{0,1\}^n$ such that $eH^T = s$ and $\mathrm{wt}(e) \leq w$

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Syndrome Decoding Problem NP-hard

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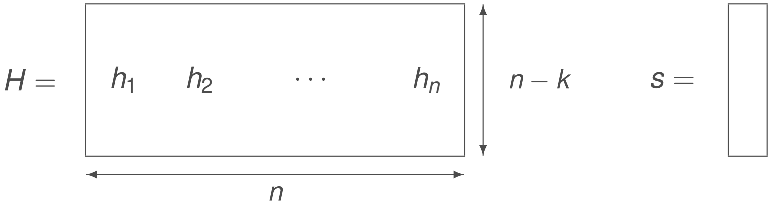
Find w columns of H adding to s (modulo 2)



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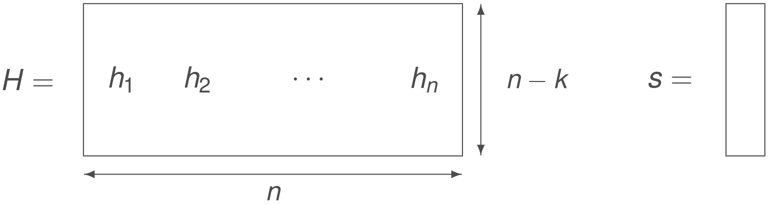

Find $1 \leq j_1 < j_2 < \cdots < j_w \leq n$ such that

$$h_{j_1} + h_{j_2} + \cdots + h_{j_w} = s$$

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$\longrightarrow$ $\dfrac{\binom{n}{w}}{2^{n-k}}$ solutions on average

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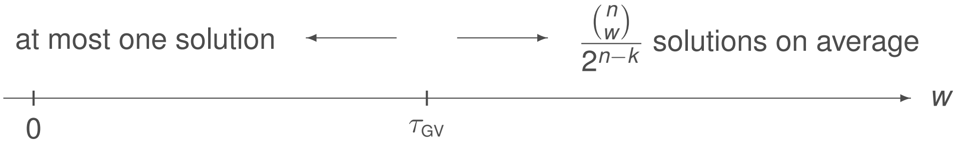

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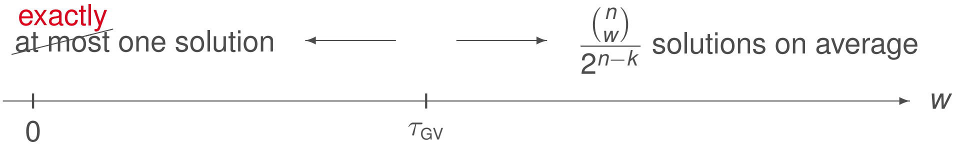

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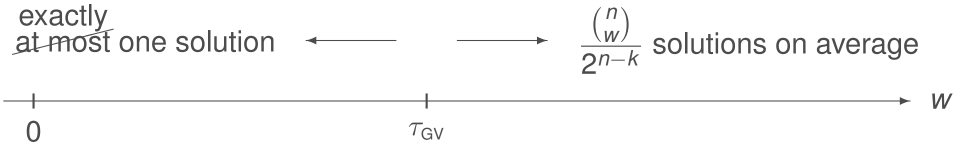

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We expect $\approx \max\left(1, \binom{n}{w}/2^{n-k}\right)$ solutions

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