Games and Markets

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One of the most basic assumptions of economic theory is that individuals take prices as given.

Quote from von Neumann and Morgenstern

"It is neither certain nor probable that a mere increase in the number of participants will always lead *in fine* to the conditions of free competition. The classical definitions of free competition all involve further postulates besides the greatness of that number. E.g., it is clear that if great groups of participants will – for any reason whatsoever – act together, then the great number of participants may not become effective ; the decisive decisions will take place directly between large "coalitions," few in number, and not between individuals, many in number acting independently. ... Any satisfactory theory of the "limiting transition" from small numbers of participants to large numbers will have to explain under what circumstances such big coalitions will or will not be formed." (page 15).

The conjecture that at individuals can do no better by forming coalitions than by taking prices as given has been shown in a number of contexts, starting with Edgeworth (1984).

Pure exchange economy and the Edgeworth box

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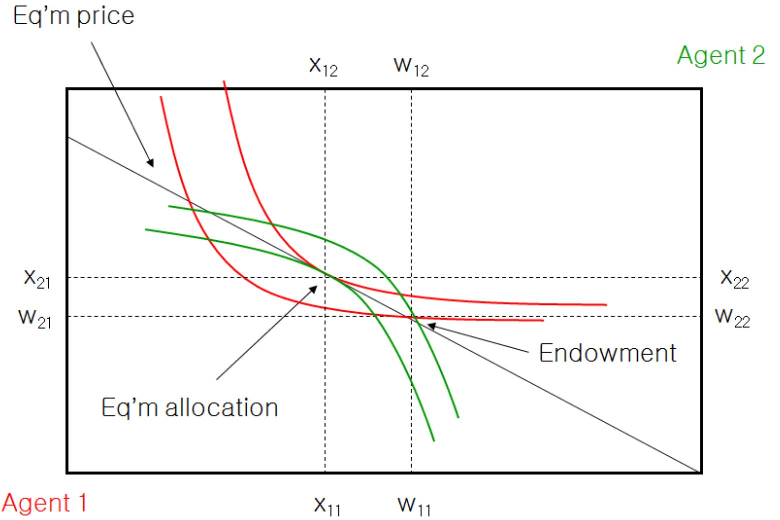
Consider a pure exchange economy with 2 consumers and 2 commodities.
— Consumer i has endowment (w_{1i}, w_{2i}) and a utility function $u_i(x_{1i}, x_{2i})$.
— The total endowment is $(\overline{w}_1, \overline{w}_2) = (w_{11} + w_{21}, w_{12} + w_{22})$.
— An *allocation* $x = (x_1, x_2) = ((x_{11}, x_{21}), (x_{12}, x_{22})) \in R^4$ is feasible if

$x_{l,1} + x_{l,2} \leq \overline{w}_l$ for $l = 1, 2$ (free dispoal with "$<$", nonwasteful with "=

— Note that the endowment is a feasible allocation.
— Under price system $p = (p_1, p_2)$, consumer i's budget constraint is

$$p \cdot x_i \leq p \cdot \overline{w}_i.$$

The following figure illustrates the Edgeworth box of such an economy.



Note Pareto optimal allocations, price line yielding the budget line, the equilibrium and the contract curve.

A *Walraisan* (or *competitive*) *equilibrium* for a pure exchange economy is a price vector p and a feasible allocation x such that $x = (x_1(p, p \cdot w_1), x_2(p, p \cdot w_2))$ where $x_i(p, p \cdot w_i)$ maximizes i's utility subject to his budget constraint, given by

$$p \cdot x_i(p, p \cdot w_i) = p \cdot w_1.$$

At a Walrasian equilibrium, under the standard assumptions of interiority of the endowment $(w_i >> 0)$, convexity of preferences, and monotonicity ("more is better") the two consumers' offer curves intersect and their indifference curves are tangent.

Sufficient conditions for existence of equilibrium are monotonicity, convexity of preferences, and $w_i >> 0$.1

A feasible allocation x is *Pareto optimal* (or Pareto efficient) if there is no other feasible allocation x' such that $x'_i \succeq_i x_i$ for all i, and $x'_i \succ_i x_i$ for some i.

A feasible allocation x is in the *core of the economy* if no group of consumers can do better for themselves using only their own resources.

But how can we address the von -Neumann-Morgenstern question more generally? Commodities may be indivisible, individuals may jointly consume commodities (swimming pools for example), they may care about whom they consume with, and so on. We will represent ecnomies by cooperative games.

Games with side payments

Constructing a game (TU) from an economy

To illustrate: Suppose that in addition to the two commodities there is money (which can be consumed in any amount) and utility functions are of the form

$$u_i(x_{1i}, x_{2i}; M_i) = M_i + h_i(x_{1i}, x_{2i})$$

where h_i satisfies standard properties of a utility function for private goods. Also suppose that there is a finite set N of consumers.

Given a subset $S \subset N$, define

$$v(S) = \max \sum_{i \in S} u_i(x_{1i}, x_{2i}; M_i)$$

subject to the conditions that

$$\sum_{i \in S} (x_{1i}, x_{2i}) = \sum_{i \in S} (w_{1i}, w_{2i})$$

and

$$\sum_{i \in S} M_i = 0.$$

Define $v(\varnothing) = 0$.

A *payoff vector* for a game (N, v) is a vector $u \in \mathbb{R}^N$.
Given $\varepsilon \geq 0$, the ε-*core* of the game is the set of payoff vectors u with the property that

$$\sum_{i \in S} u_i \geq v(S) - \varepsilon |S| \text{ for all } S \subset N$$

and

$$\sum_{i \in N} u_i \leq v(N).$$

Relate core to contract curve.

Given a game (N, v) two players i and j are *substitutes* if

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

for all $S \subset N$ with $i \notin S$ and $j \notin S$.
Call all players who are substitutes players *of the same type.*

Preview – under “small group effectiveness” *any* economy representable as a cooperative game where every (or most) players have many substitutes is approximated by an economy (a market). One possible approximating market is where the player types are the types of commodities. Approximate cores are non-empty and converge to payoffs that treat all players of the same type identically (equal-treatment payoffs).

Pregames

T – a positive integer, interpreted as a number of player types (attributes or characteristics).

Example: $T = 2$, the types are males and females

$f = (f_1, \cdots, f_T) \in \mathbb{Z}_+^T$ – a *profile* describing a group of players by the numbers of players of each type in the group.

Example: $f = (3, 2)$, say 3 boys and 2 girls

A *subprofile of a profile* f is a profile f' satisfying $f' \leq f$.

Example: $f' = (2, 2) \leq (3, 2)$

Pregames

The worth function of a pregame

Ψ – a function from the set of profiles $\mathbb{Z}_+^T$ to $\mathbb{R}_+$ with $\Psi(0) = 0$.

Example: $\Psi(f_1, f_2) = \min\{f_1, f_2\}$.
Type 1 players each own a LH glove, type 2 players each own a RH glove. A pair of gloves is worth a dollar and singleton gloves are worth 0.

(T, Ψ) – a *pregame* with *worth function* Ψ.

Example: Take as given two continuous, concave utility functions u_1 and u_2 and two endowment vectors e^1 and e^2, both in $\mathbb{R}^L_+$.
Type 1 players have utility function u_1 and endowment e^1 and similarly for type 2 players.
Given a profile $f \in \mathbb{Z}^2_+$ define

$$\Psi(f) = \max_{x,y} \sum (f_1 u_1(x) + f_2 u_1(y))$$

subject to the conditions that

$$f_1 x + f_2 y = f_1 e^1 + f_2 e^2$$

$$x, y \geq 0.$$

Define Ψ^*, the *superadditive cover* of Ψ, by

$$\Psi^*(f) \stackrel{\text{def}}{=} \max \sum_k \Psi(f^k),$$

where the maximum is taken over the set of all partitions $\{f^k\}$ of f $(\sum_k f^k = f)$.

If the worth functions Ψ and Ψ^* are equal then (T, Ψ) is *superadditive*.

Pregames

Example of a pregame and its superadditive cover

Example: $T = 1$, $\Psi(m) = 6$ if $m = 2$, otherwise $\Psi(m) = 0$.

Then:
$\Psi^*(m) = 3m$ if m is an even integer and $\Psi^*(m) = 3(m-1)$ otherwise.

Games derived from pregames

A *game determined by the pregame* (T, Ψ) is a pair $[n; (T, \Psi)]$ where n is a profile.

A *subgame* of a game $[n; (T, \Psi)]$ is a pair $[f; (T, \Psi)]$ where f is a subprofile of n.

Games derived from pregames

Deriving games in the usual form

With $[n; (T, \Psi)]$ we associate a game (N, v): Let

$$N = \{(t, q) : t = 1, \cdots, T \text{ and } q = 1, \cdots, n_t\}$$

be a *player set* for the game.

For each $S \subset N$ define $prof(S) \in \mathbb{Z}_+^T$, by its components

$$prof(S)_t \overset{\text{def}}{=} S \cap \{(t', q) : t' = t \text{ and } q = 1, \cdots, n_t\}$$

and define

$$v(S) \overset{\text{def}}{=} \Psi(prof(S)).$$

Then the pair (N, ν) satisfies the usual definition of a game with side payments.

Games and pregames

Deriving games in the usual form

For any $S \subset N$, define

$$v^*(S) \stackrel{\text{def}}{=} \Psi^*(\mathit{prof}(S)).$$

The game (N, ν^*) is the *superadditive cover of* (N, ν).

A pregame (T, Ψ) satisfies **small group effectiveness** if, for each $\varepsilon > 0$, there is an integer $\eta_1(\varepsilon)$ such that for every profile f there is a partition $\{f^k\}$ of f satisfying:

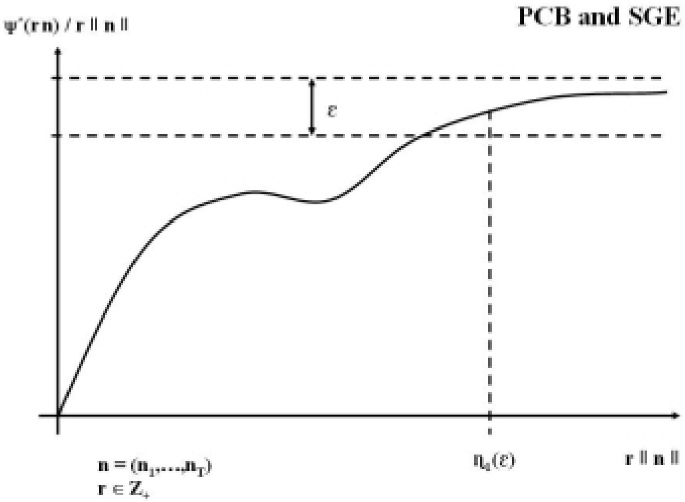
$$\|f^k\| \leq \eta_1(\epsilon) \text{ for each subprofile } f^k, \text{ and}$$

$$\Psi^*(f) - \Sigma_k \Psi(f^k) \leq \varepsilon\|f\|;$$

given $\varepsilon > 0$ there is a group/coalition size $\eta_3(\varepsilon)$ such that within ε per capita of the gains to group formation can be realized by the collective activities of groups containing no more than $\eta_1(\varepsilon)$ players.

Games and pregames

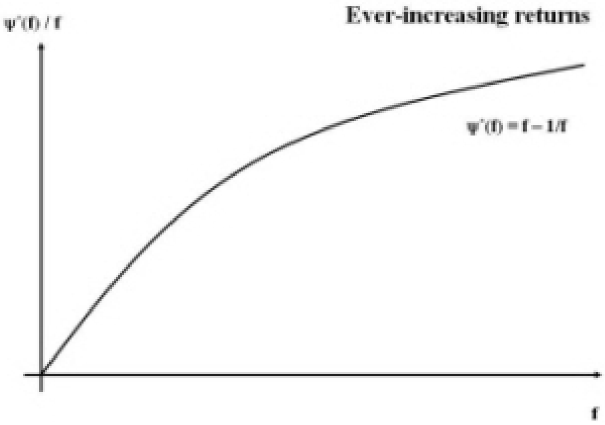
Small group effectiveness



Games and pregames

Small group effectiveness

Examples: (1) Matching games, (2) assignment games (3) games generated by economies where all players have the same concave utility function or, for a more specific example, (4) $T = 1$, $\Psi^*(f) = f - \frac{1}{f}$



Games and pregames

Per capita boundedness (PCB) dictates that:

$$PCB: \quad \sup_{f \in \mathbb{Z}_+^T} \frac{\Psi(f)}{\|f\|} \text{ is finite}$$

or equivalently,

$$\sup_{f \in \mathbb{Z}_+^T} \frac{\Psi^*(f)}{\|f\|} \text{ is finite.}$$

Games and pregames

MW 1994, Econometrica, Theorem 4. *With 'thickness,' SGE=PCB.*

(1) Let (T, Ψ) be a pregame satisfying SGE. Then the pregame satisfies PCB.

(2) Let (T, Ψ) be a pregame satisfying PCB. Then given any positive real number ρ, construct a new pregame (T, Ψ_ρ) with the domain of Ψ_ρ is restricted to profiles f where, for each $t = 1, \cdots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$. Then (T, Ψ_ρ) satisfies SGE on its domain.

Generating a limiting market utility function from a pregame

Let (T, Ψ) satisfy SGE. For each x in $\mathbb{R}_+^T$ define

$$U(x) \stackrel{\text{def}}{=} \|x\| \lim_{\nu \to \infty} \frac{\Psi^*(f^\nu)}{\|f^\nu\|}$$

where the sequence $\{f^\nu\}$ satisfies

$$\lim_{\nu \to \infty} \frac{f^\nu}{\|f^\nu\|} = \frac{x}{\|x\|}$$

and

$$\|f^\nu\| \to \infty.$$

Proposition (MW 1988, 1994, Lemma 2). Assume SGE holds. Then for any $x \in \mathbb{R}_+^T$ the limit above exists and $U(\cdot)$ is a well-defined concave function.

The pregame: There are two types of players - cooks and helpers. Suppose a banquet is worth 10 dollars and unemployment insurance is worth 1 dollar.

(a) 1 cook and 2 helpers can make a banquet; $\Psi(1,2) = 10$

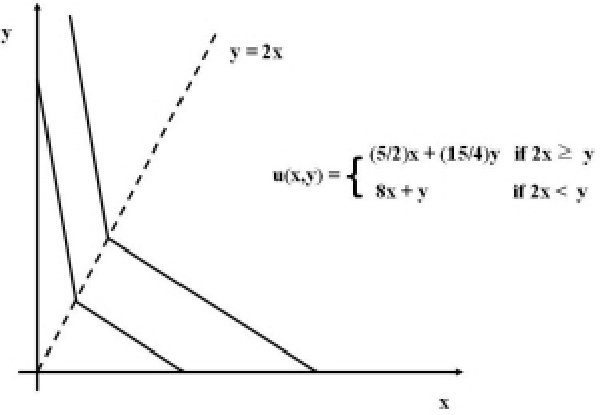
(b) 4 cooks alone can make a banquet (cooks are not very efficient as helpers); $\Psi(4,0) = 10$

(c) A helper can find his way to the unemployment insurance office and collect unemployment benefits. $\Psi(0,1) = 10$

(d) All other groups in the kitchen are useless: $\Psi(m_1, m_2) = 0$ otherwise.

The utility function: $u(x,y) = \left[\begin{array}{ll} \frac{5x}{2} + \frac{15y}{4} & \text{if } 2x \geqslant y \\ 8x + y & \text{if } 2x < y \end{array} \right]$

Indifference curves



Generating a limiting market utility function from a pregame

Remark: With just PCB, U is not well defined at the "boundaries". Let $T = 2$ and define

$$\Psi^*(k,n) = \begin{cases} k+n & \text{when } k > 0 \\ 0 & \text{otherwise} \end{cases}$$

Ψ^* satisfies PCB but not SGE.

Let $\{x^\nu\}$ and $\{y^\nu\}$ be given by $x^\nu = (1, \nu - 1)$ and $y^\nu = (0, \nu)$. Then

$$\lim_{\nu\to\infty} \frac{1}{\|x^\nu\|} x^\nu = \lim_{\nu\to\infty} \frac{1}{\|y^\nu\|} y^\nu = (0,1)$$

but

$$\lim_{n\to\infty} \frac{\Psi^*(x^\nu)}{\|x^\nu\|} = 1 \text{ but } \lim_{\nu\to\infty} \frac{\Psi^*(y^\nu)}{\|y^\nu\|} = 0$$

SGE rules out this difficulty.

Small group negligibility

A pregame (T, Ψ) satisfies *small group negligibility* if it satisfies PCB and if, for any sequence of profiles $\{f^\nu\}_\nu$ where

$$||f^\nu||_1 \to \infty \text{ as } \nu \to \infty,$$
$$\mathit{support}(f^\nu) = \mathit{support}(f^{\nu'}) \text{ for all } \nu \text{ and } \nu' \text{ and}$$
$$\lim \tfrac{1}{\|f^\nu\|_1} f^\nu \text{ and } \lim_{\nu\to\infty} \tfrac{\Psi(f^\nu)}{\|f^\nu\|_1} \text{ both exist,}$$

then, for any sequence of profiles $\{\ell^\nu\}$ with

$$\lim_{\nu\to\infty} \frac{\|\ell^\nu\|_1}{\|f^\nu\|_1} = 0,$$

it holds that

$$\lim_{\nu\to\infty} \tfrac{\Psi(f^\nu+\ell^\nu)}{\|f^\nu+\ell^\nu\|_1} \text{ exists, and}$$

$$\lim_{\nu\to\infty} \tfrac{\Psi(f^\nu+\ell^\nu)}{\|f^\nu+\ell^\nu\|_1} = \lim_{\nu\to\infty} \tfrac{\Psi(f^\nu)}{\|f^\nu\|_1}.$$

Quote from von Neumann and Morgenstern

"Any satisfactory theory of the "limiting transition" from small numbers of participants to large numbers will have to explain under what circumstances such big (effective) coalitions will or will not be formed – i.e. when the large numbers of participants will become effective and lead to more or less free competition. Answering this question is, we think, the real challenge to any theory of free competition" (page 15).

(Note some other works but from different directions.)

Generating a limiting market utility function from a pregame

From concavity U, for any player profile $n = (n_1, ..., n_T)$, $[n; (T, U)]$ is a totally balanced game.

Implications:

- Given any population n, which could represent a game with a continuum of players, where n_t is the measure of players of type t in the game, or a finite game, where n_t is the number of players of type t the game has a nonempty core.
- Assuming a continuum of players, both the Aumann-core and the f-core of the game are nonempty (and equal). [M. Kaneko and MW.] The Aumann-core allows improvement by only coalitions of positive measure. The f-core allows improvement by only finite coalitions.

Generating a limiting market utility function from a pregame

Implications: (continued)

- The game is representable as a market. One possible market is where each player owns one unit of a commodity, his type and all players have the same utility function.
- For any large finite game the game is approximately a market game and thus approximately representable as a game of flow.

Cores of games and markets

Take as given a game $[n, (T, \Psi)]$ and induced game (N, v) where $N = \{(t, q) : t = 1, ..., T \text{ and } q = 1, ..., n_t\}$ – naming the players.

A payoff vector $x \in \mathbb{R}^N$ is *feasible* if

$$x(N) \leq v^*(N) = \Psi^*(prof(N))$$

and satisfies *equal-treatment* if $x^{tq} = x^{tq'}$ for all $q, q' \in \{1, \cdots, n_t\}$ and for each $t = 1, \cdots, T$.

For any $\varepsilon \geq 0$ a payoff vector x is in the ε*-core of the derived game with player set* N if $x(N) \leq \Psi^*(n)$ and if, for all nonempty subsets $S \subset N$ (coalitions),

$$x(S) \geq \Psi^*(prof(S)) - \varepsilon \|prof(S)\|.$$

An *equal-treatment payoff vector* is a point $\overline{x}$ in $\mathbb{R}^T$.
$\overline{x} \in \mathbb{R}^T$ is feasible if:

$$\Psi^*(n) \geq \overline{x} \cdot n.$$

$\overline{x}$ is in the *equal-treatment ε-core* of $[n; (T, \Psi)]$ or simply "in the ε-core" if $\overline{x}$ is feasible for $[n; (T, \Psi)]$ and

$$\Psi(s) \leq \overline{x} \cdot s + \varepsilon \|s\| \text{ for all subprofiles } s \text{ of } n.$$

When $\varepsilon = 0$, we call the ε-core simply the *core*.

Proposition (MW): *Nonemptiness of approximate cores*. Let (T, Ψ) be a pregame satisfying SGE. Let ε be a positive real number. Then there is an integer $\eta_1(\varepsilon)$ such that any game $[n; (T, \Psi)]$ with $\|n\| \geq \eta_1(\varepsilon)$ has a nonempty ε-core.

[Note that no assumption of superadditivity is required but only because our definition of feasibility is in terms of the superadditive cover.]

Given a vector $f \in \mathbb{Z}_+^T$, let $\Pi(f) = [u \in \mathbb{R}_+^T : u(f) = U(f)$ and for all $s \leq f$, $s \in \mathbb{R}_+^T$ it holds that $u \cdot s \geq U(s)$.

Theorem: *Uniform closeness of (equal-treatment) approximate cores to the core of the limit game.* Let (T, Ψ) be a pregame satisfying SGE and let $\Pi(\cdot)$ be as defined above. Let $\delta > 0$ and $\rho > 0$ be positive real numbers. Then there is a real number ε^* with $0 \leq \varepsilon^*$ and an integer $\eta_0(\delta, \rho, \varepsilon^*)$ with the following property: for each positive $\varepsilon \in (0, \varepsilon^*]$ and each game $[n; (T, \Psi)]$ with $\|n\| > \eta_0(\delta, \rho, \varepsilon^*)$ and $\frac{n_t}{\|n\|} \geq \rho$ for each $t = 1, \cdots, T$, if $C(n; \varepsilon)$ is nonempty then

$$\mathrm{dist}[C(n; \varepsilon), \Pi(f)] < \delta$$

where 'dist' is the Hausdorf distance with respect to the sum norm on $\mathbb{R}^T$ and $C(n; \varepsilon)$ is the equal-treatment ε-core of the game $[n, (T, \Psi)]$.

Example. (*Without thickness, PCB does not imply equal treatment,* **even** *of players of abundant types.*) Let (T, Ψ) be a pregame where $T = 2$ and the worth $\Psi(n)$ to any profile n is given by:

$$\Psi(n) = \begin{cases} n(\omega_1) + n(\omega_2) & \text{if } n(\omega_1) > 0,\ n(\omega_2) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now consider a sequence of games (N^ν, v^ν) where the profile of N^ν is denoted by n^ν and satisfies $n^\nu(\omega_1) = 1,\ n^\nu(\omega_2) = \nu$.
Consider a payoff vector $x^\nu \in \mathbb{R}^{N^\nu}$ assigning $x^\nu_{2q} = \frac{q}{\nu}$ to the q^{th} player of type 2, $q = 1, ..., \nu$, and assigning $1 + \nu - \sum_q x^\nu_{2q}$ to the one player of type 1.
Then, for any ν, x^ν is in the core of the game (N^ν, v^ν).
With some additional work, the same conclusion can be obtained for approximate cores.

Market example

Suppose the following primitives are given: K types of commodities, T players types, an endowment vector e^t and a utility function u_t for each $t = 1, ..., T$.

Given a profile n, define $\Psi(n)$ as the maximal total utility of a group of players.

Each player of type t will have the endowments e^t and utility function u_t.

Given any profile n, the all players have the same (quasi-linear) utility function u, and there are K commodities.

The endowment of player t is e^t.

Assume Ψ satisfies samll group effectiveness.

Market example

The game generated by the "limiting game", in which all players have the same concave utility function U , is totally balanced (and thus representable as a game of flow).

Suppose that a vector of utilities $u = (u_1, ..., u_T)$ is in the core of the game generated by the limiting economy.

Then there is a vector p (prices) such that

$$\begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_T \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & & e_{1K} \\ . & . & . & . \\ & & & \\ e_{T1} & & & e_{TK} \end{bmatrix} \begin{bmatrix} p_1 \\ . \\ . \\ . \\ p_K \end{bmatrix}$$

This holds for limiting market games. For large finite games, approximate cores are close to cores of limiting market games – so there exists approximate competitive equilibria in finite games with many players.

Conditions

A pregame (T, Ψ) satisfies *small group negligibility* if it satisfies PCB and if, for any sequence of profiles $\{f^\nu\}_\nu$ where

$$||f^\nu||_1 \to \infty \text{ as } \nu \to \infty,$$
$$\mathit{support}(f^\nu) = \mathit{support}(f^{\nu'}) \text{ for all } \nu \text{ and } \nu' \text{ and}$$
$$\lim \tfrac{1}{\|f^\nu\|_1} f^\nu \text{ and } \lim_{\nu\to\infty} \tfrac{\Psi(f^\nu)}{\|f^\nu\|_1} \text{ both exist,}$$

then, for any sequence of profiles $\{\ell^\nu\}$ with

$$\lim_{\nu\to\infty} \frac{\|\ell^\nu\|_1}{\|f^\nu\|_1} = 0,$$

it holds that

$$\lim_{\nu\to\infty} \tfrac{\Psi(f^\nu+\ell^\nu)}{\|f^\nu+\ell^\nu\|_1} \text{ exists, and}$$

$$\lim_{\nu\to\infty} \tfrac{\Psi(f^\nu+\ell^\nu)}{\|f^\nu+\ell^\nu\|_1} = \lim_{\nu\to\infty} \tfrac{\Psi(f^\nu)}{\|f^\nu\|_1}.$$

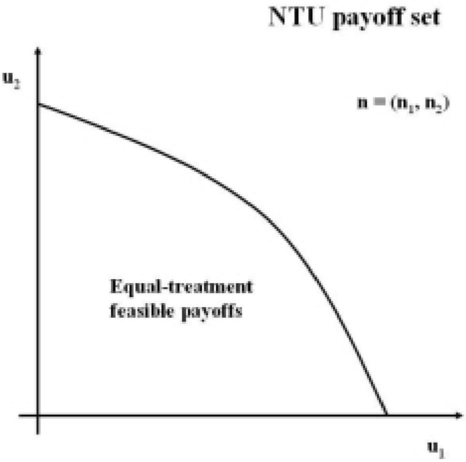
The property of small group negligibility appears quite mild.

It simply ensures that a small group of possibly distinct player types cannot significantly affect per capita payoffs of large player sets.

Theorem (MW)**: (***Equivalence of small group effectiveness, SGE, and small group negligibility, SGN***):** Let (Ω, Ψ) be a pregame. Then (Ω, Ψ) satisfies SGE if and only if (Ω, Ψ) satisfies SGN.

NTU games and pregames

NTU games



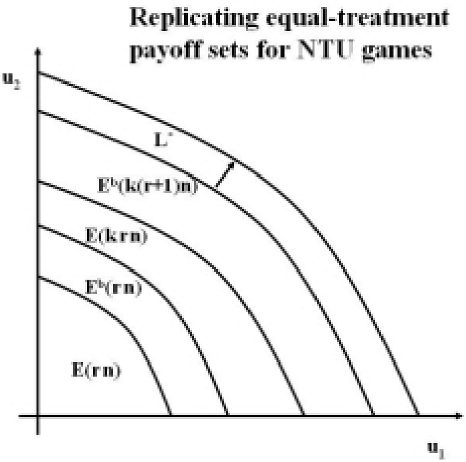
NTU games and pregames

Replicating equal treatment payoff sets for NTU games

MW 1983

Take a profile $n = (n_1, ..., n_T)$ as given.
Replicate the profile – take the profile $rn = (rn_1, ..., rn_T)$ and consider the sets of equal treatment payoffs.
Let $E(rn)$ denote the set of equal treatment payoffs for the r^{th} replica game.
From superadditivity $E(rn) \subset E(krn)$ for any positive integer k.
We can also define "the balanced cover game" for each game – this is the game with the "smallest" payoff sets containing the original payoff sets and having a nonempty core. Let $E^b(rn)$ denote the equal treatment payoff set for the balanced cover of the r^{th} replica game.
From a PCB assumption, there is a limiting set.

NTU games and pregames



NTU games and pregames

Implications

MW 1983

We can now get all the results of the prior section (except for the limiting utility function). In particular, (in research with N. Allouch)

Implications:

- Given any population n, which could represent a game with a continuum of players, where n_t is the measure of players of type t in the game, or a finite game, where n_t is the number of players of type t the game has a nonempty core.
- Assuming a continuum of players, both the Aumann-core and the f-core of the game are nonempty (and equal). [M. Kaneko and MW.] The Aumann-core allows improvement by only coalitions of positive measure. The f-core allows improvement by only finite coalitions.
- Approximate cores of large finite games are nonempty and "close" to cores of the limiting game.

Thanks

This research, for me, began with working papers on TU games and MW (1983). Since then I have collaborated with M. Shubik, M. Kaneko, W. Zame, A. Kovalenkov and N. Allouch. Thanks to them for their collaborations on parts of this research! Thanks to Sunghoon Hong for the figures.

And THANKS TO YOU for LISTENING!